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Short Note

A primal formulation for the Helmholtz decomposition

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Abstract

In 1999, Jean-Paul Caltagirone and Jérôme Breil have developed in their paper [Caltagirone, J. Breil, Sur une méthode de projection vectorielle pour la résolution des équations de Navier–Stokes, C.R. Acad. Sci. Paris 327(Série II b) (1999) 1179–1184] a new method to compute a divergence-free velocity. They have used the **grad**(div) operator to extract the solenoidal part of a given vector field. In this contribution we explain how this method can be considered as a real Helmholtz decomposition and we present a stable approximation in the framework of spectral methods. Numerical results are presented to illustrate the efficiency of this approach.

Keywords: **grad**(div) operator; Stable approximation; Helmholtz decomposition

1. Introduction

The approximation of the **grad**(div) operator pervades many applied physics domains. Besides the ideal ocean wave problem without Coriolis force and no friction [15], it arises in the Maxwell equations [11] and in the Navier–Stokes equations for fluid flow problems when using a penalty formulation for the incompressibility condition [14]. The problem also arises in the ideal linear magneto hydrodynamics equations when computing the stability behavior of a fusion plasma device [16]. Another original application of this operator was introduced by J.P. Caltagirone and J. Breil in their paper [13] where they used this operator to extract from a given velocity field its solenoidal part. These authors had christened it *vector projection* which consists in solving the following problem: Let $\mathbf{u}^*$ be a non-divergence free velocity field, find a couple of vector fields $(\mathbf{u}, \mathbf{v})$ such that

$$-\boldsymbol{\nabla}(\nabla \cdot \mathbf{u}) = \boldsymbol{\nabla}(\nabla \cdot \mathbf{u}^*), \quad \text{in } \Omega, \tag{1.1}$$
$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \tag{1.2}$$
$$\mathbf{v} = \mathbf{u} + \mathbf{u}^*, \quad \text{in } \Omega, \tag{1.3}$$

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where $\mathbf{v}$ and $\mathbf{u}$ are respectively divergence-free and curl-free. Here $\Omega \subset \mathbb{R}^d (d = 2, 3)$ is a simply connected and bounded domain with Lipschitzian border. $\mathbf{n}$ denotes the outer unit normal along the boundary.

The objective of this note is on the one hand to explain how the previous system can be considered as a Helmholtz decomposition step and on the other hand to present a stable discretization in the framework of spectral methods. We end this note by presenting a relevant numerical experiment.

Some notations – The symbol $L^2(\Omega)$ stands for the usual Lebesgue space and $H^1(\Omega)$, the Sobolev space, involves all the functions that are, together with their gradient, in $L^2(\Omega)$. The $\mathscr{C}(\Omega)$ denotes the space of continuous functions defined in Ω.

2. Continuous problems and their variational formulations

In order to write the continuous problem in its variational form we introduce the relevant spaces of functions.

Let $H(\mathrm{div},\Omega)$ denote the space (see [12])

$$H(\mathrm{div}, \Omega) = \{\mathbf{w} \in (L^2(\Omega))^d; \mathrm{div}\,\mathbf{w} \in L^2(\Omega)\},$$

endowed with the natural norm

$$\|\mathbf{w}\|_{H(\mathrm{div},\Omega)} = (\|\mathbf{w}\|^2_{(L^2(\Omega))^d} + \|\mathrm{div}\,\mathbf{w}\|^2_{L^2(\Omega)})^{1/2}.$$

The continuous problem we consider reads: *Find* $\mathbf{u}$ *in* $H(\mathrm{div}, \Omega)$ it such that:

$$-\nabla(\nabla \cdot \mathbf{u}) = \mathbf{f}, \quad \text{in } \Omega, \tag{2.4}$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \tag{2.5}$$

where $\mathbf{f}$ is a given data.

Since curl ($\mathbf{grad}\,\cdot) \equiv \mathbf{0}$ we notice that a necessary condition for the existence of a solution to problems (2.4) and (2.5) is that curl $\mathbf{f} = \mathbf{0}$ and by consequence we can state the existence of a function $\varphi(x, y)$ such that

$$\mathbf{f} = \mathbf{grad}\,\varphi.$$

This leads to restate the basic problem as: *For a given* $\varphi \in L^2_0(\Omega)$, *find* $\mathbf{u} \in H(\mathrm{div},\Omega)$ *such that*

$$-\nabla(\nabla \cdot \mathbf{u}) = \nabla\varphi, \quad \text{in } \Omega, \tag{2.6}$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \tag{2.7}$$

where $L^2_0(\Omega)$ denotes the $L^2(\Omega)$ subspace of functions having zero average values. This formulation is equivalent to the dual one that reads: *For a given* $\varphi \in L^2_0(\Omega)$ *find* $\mathbf{u} \in X(\Omega)$ *and* $\psi \in L^2_0(\Omega)$ *such that*:

$$\mathbf{u} - \nabla\psi = 0, \quad \text{in } \Omega, \tag{2.8}$$

$$-\nabla \cdot \mathbf{u} = \varphi, \quad \text{in } \Omega, \tag{2.9}$$

$$\mathbf{u} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega, \tag{2.10}$$

where

$$X(\Omega) = \{\mathbf{w} \in H(\mathrm{div}, \Omega); \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

This dual formulation can be rewritten as a classical Helmholtz decomposition, indeed: Let $\mathbf{u}^*$ and $\mathbf{v}$ be two vector fields such that $\nabla \cdot \mathbf{u}^* = \varphi$, $\mathbf{u}^* \cdot \mathbf{n} = 0$ and $\mathbf{v} = \mathbf{u} + \mathbf{u}^*$. The problem (2.8)–(2.10) then becomes: *Find* $\mathbf{v} \in X(\Omega)$ *and* ψ in $L^2_0(\Omega)$ *such that*

$$\mathbf{v} - \nabla\psi = \mathbf{u}^*, \quad \text{in } \Omega, \tag{2.11}$$

$$\nabla \cdot \mathbf{v} = 0, \quad \text{in } \Omega, \tag{2.12}$$

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega. \tag{2.13}$$

Consequently the Helmholtz decomposition of the vector field $\mathbf{u}^*$ can be achieved using either the primal formulation (1.1)–(1.3) or its equivalent dual one (2.11)–(2.13).

Variational formulation – One obtains the primal variational formulation of problem (2.6) and (2.7) by taking the inner product of 2.6 with any $\mathbf{w} \in X(\Omega)$. After integration by parts the problem becomes: *Find* $\mathbf{u} \in X(\Omega)$ *such that*:

$$\int_\Omega \nabla \cdot \mathbf{u} \nabla \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} = -\int_\Omega \varphi \nabla \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} \quad \forall \mathbf{w} \in X(\Omega). \tag{2.14}$$

We do likewise to write the variational formulation of the dual problem (2.11)–(2.13) and we get : *Find* $(\mathbf{v}, \psi) \in X(\Omega) \times L_0^2(\Omega)$ *such that*

$$\int_\Omega \mathbf{v} \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} + b(\mathbf{w}, \psi) = \int_\Omega \mathbf{u}^* \cdot \mathbf{w} \, \mathrm{d}\mathbf{x}, \quad \forall \mathbf{w} \in X(\Omega), \tag{2.15}$$

$$b(\mathbf{v}, q) = 0, \quad \forall q \in L_0^2(\Omega), \tag{2.16}$$

where the bilinear form $b(\mathbf{w}, q)$ defined over $X(\Omega) \times L_0^2(\Omega)$ is given by:

$$b(\mathbf{w}, q) = \int_\Omega (\operatorname{div} \mathbf{w}) q \, \mathrm{d}\mathbf{x}.$$

The variational formulation (2.15) and (2.16) is that of a saddle-point problem. One checks easily that the bilinear form $b(\cdot,\cdot)$ satisfies an inf–sup condition with a positive constant β (see [12,17]) such that:

$$\sup_{\mathbf{w} \in X(\Omega)} \frac{b(\mathbf{w}, q)}{\|\mathbf{w}\|_{H(\operatorname{div},\Omega)}} \geqslant \beta \|q\|_{L^2(\Omega)}, \quad \forall q \in L_0^2(\Omega).$$

3. Stable discretization

Providing a stable approximation of the primal and dual problem is a difficult task. Consequently we supply a non-exhaustive list of references dedicated to this question in the framework of spectral methods [3] and finite element approximation [5–10]. Concerning the finite volume context one can see [13].

The equivalence between the two variational formulations (2.14) and (2.15), (2.16) brings us to propose a stable discretization. For the sake of clarity we will suppose from now that $d = 2$ and Ω is a square $]-1,+1[^2$ and we limit ourself to the spectral approximation.

Let $\mathbb{P}_N(\Omega)$ represent the set of all polynomials of degree less or equal to N with respect to each space variables. We denote $X_N(\Omega)$ the velocity space that is a subspace of $(\mathbb{P}_N(\Omega) \times \mathbb{P}_N(\Omega)) \cap X(\Omega)$. The finite dimensional primal variational of (2.14) writes: *Find* $\mathbf{u}_N \in X_N(\Omega)$ *such that*

$$\int_\Omega \nabla \cdot \mathbf{u}_N \nabla \cdot \mathbf{w}_N \, \mathrm{d}\mathbf{x} = -\int_\Omega \varphi_N \operatorname{div} \mathbf{w}_N \, \mathrm{d}\mathbf{x}, \quad \forall \mathbf{w}_N \in X_N(\Omega). \tag{3.17}$$

For any integer N, $\mathbb{P}_N(\Lambda)$ represents the set of all polynomials of degree $\leqslant N$ on $\Lambda =]-1,+1[$ and $\mathbb{P}_N^0(\Lambda)$ coincides with $\mathbb{P}_N(\Lambda) \cap H_0^1(\Lambda)$. Since spectral methods make an extensive use of Gaussian quadrature rules, we recall the basic properties of the Gauss–Legendre (GL) and Gauss–Lobatto–Legendre (GLL) quadratures schemes.

- GL quadrature rule: A unique set of N nodes τ_j and associated coefficients ω_j exist ($\tau_j \in \Lambda, \omega_j > 0$, real, $1 \leqslant j \leqslant N$) such that

$$\forall \Phi \in \mathbb{P}_{2N-1}(\Lambda), \quad \int_{-1}^{1} \Phi(\tau) \, \mathrm{d}\tau = \sum_{j=1}^{N} \Phi(\tau_j) \omega_j.$$

The nodes τ_j $(1 \leqslant j \leqslant N)$ are solutions to $L_N = 0$ where L_N denotes the Legendre polynomial of degree N.

- GLL quadrature rule: Let $\xi_0 = -1$ and $\xi_N = 1$. A unique set of nodes $\xi_j \in \Lambda, (1 \leqslant j \leqslant N-1)$ and $(N+1)$ real, positive coefficients ρ_j exist, such that

$$\forall \Phi \in \mathbb{P}_{2N-1}(\Lambda), \quad \int_{-1}^{1} \Phi(\xi)\,\mathrm{d}\xi = \sum_{j=0}^{N} \Phi(\xi_j)\rho_j.$$

The nodes ξ_j $(0 \leqslant j \leqslant N)$ are the solutions to $(1-\xi^2)L'_N = 0$.

We also introduce the canonical polynomial interpolation basis $h_i(x) \in \mathbb{P}_N(\Lambda)$ built on the GLL nodes and given by the relationships:

$$h_i(x) = -\frac{1}{N(N+1)}\frac{1}{L_N(\xi_i)}\frac{(1-x^2)L'_N(x)}{(x-\xi_i)}, \quad -1 \leqslant x \leqslant +1, \ \ 0 \leqslant i \leqslant N, \tag{3.18}$$

with the elementary cardinality property

$$h_i(\xi_j) = \delta_{ij}, \quad 0 \leqslant i, \ \ j \leqslant N, \tag{3.19}$$

where δ_{ij} is Kronecker's delta symbol. We further introduce the canonical polynomial interpolation basis $\tilde{h}_j(x) \in \mathbb{P}_{N-1}(\Lambda)$ built on GL nodes

$$\tilde{h}_j(x) = \frac{1}{L'_N(\zeta_j)}\frac{L_N(x)}{(x-\zeta_j)}, \quad -1 \leqslant x \leqslant +1, \ \ 1 \leqslant j \leqslant N. \tag{3.20}$$

The functions (3.20) satisfy the same property (3.19) with respect to the GL nodes ζ_j.

In [3] we have shown that a stable and optimal spectral scheme exists to solve (3.17). It uses a GLL/GL–GL/GLL mesh for the two components of the velocity field and corresponds to: *Find* $\mathbf{u}_N \in X_N(\Omega)$ *such that*

$$(\nabla\cdot\mathbf{u}_N, \nabla\cdot\mathbf{w}_N)_N^G = -(\varphi_N, \operatorname{div}\mathbf{w}_N)_N^G, \quad \forall \mathbf{w}_N \in X_N(\Omega), \tag{3.21}$$

with, for instance:

$$X_N(\Omega) = (\mathbb{P}_N^0(\Lambda) \otimes \mathbb{P}_{N-1}(\Lambda)) \times (\mathbb{P}_{N-1}(\Lambda) \otimes \mathbb{P}_N^0(\Lambda))$$

and for any couple of scalar fields p, q:

$$(p,q)_N^G = \sum_{i=1}^{N}\sum_{j=1}^{N}(pq)(\tau_i,\tau_j)\omega_i\omega_j. \tag{3.22}$$

Using the basis functions (3.18) and (3.20) the velocity components write

$$u_{xN}(x,y) = \sum_{k=1}^{N-1}\sum_{\ell=1}^{N} u_{k\ell}^x h_k(x)\tilde{h}_\ell(y), \tag{3.23}$$

$$u_{yN}(x,y) = \sum_{k=1}^{N}\sum_{\ell=1}^{N-1} u_{k\ell}^y \tilde{h}_k(x)h_\ell(y), \tag{3.24}$$

where $u_{k\ell}^x$ (resp. $u_{k\ell}^y$) denotes $u_x(\xi_k,\zeta_\ell)$ (resp. $u_y(\zeta_k,\xi_\ell)$), and the boundary conditions have been included in the expansions.

Problem (3.21) leads to an algebraic problem

$$\mathscr{K}\mathbf{u} = \mathbf{f}, \tag{3.25}$$

where $\mathscr{K}$ denotes the (symmetric) stiffness matrix. The stiffness matrix has two-by-two block structures, namely

$$\mathscr{K} = \begin{pmatrix} K_{xx} & K_{yx} \\ K_{xy} & K_{yy} \end{pmatrix},$$

where K_{xx} and K_{yy} denote the classical stiffness matrices similar to the discretization of the Laplacian operator, and K_{xy} and K_{yx} (transpose of each other) result from the weak formulation of the mixed derivatives present in the grad(div) operator. The quantities $\mathbf{u}$ and $\mathbf{f}$ represent respectively the vector with the nodal values as the unknowns and the data $-(\varphi_N, \operatorname{div}\mathbf{w}_N)_N^G$.

We now switch to the discrete form of the dual variational formulation (2.15), (2.16). As proved in [3], a stable spectral element to solve this problem is made of $X_N(\Omega)$ and $M_N(\Omega)$ respectively for the "velocity" ($\mathbf{v}$) and "pressure" (ψ) fields, where:

$$M_N(\Omega) := \mathbb{P}_{N-1}(\Omega) \cap L_0^2(\Omega).$$

Consequently, the spectral approximation of (2.15), (2.16) reads: *Find* $(\mathbf{v}_N, \psi_N) \in X_N(\Omega) \times M_N(\Omega)$ *such that*

$$(\mathbf{v}_N, \mathbf{w}_N)_N^{GL} + (\psi_N, \nabla \cdot \mathbf{w}_N)_N^G = (\mathbf{u}_N^*, \mathbf{w}_N)_N^{GL}, \quad \forall \mathbf{w}_N \in X_N(\Omega), \tag{3.26}$$

$$(\nabla \cdot \mathbf{v}_N, q_N)_N^G = 0, \quad \forall q_N \in M_N(\Omega). \tag{3.27}$$

One can easily verify (see [2,1]) that:

Lemma. *The following uniform inf–sup condition on* $b(\cdot,\cdot)$ *holds*: $\forall q_N \in M_N(\Omega)$,

$$\sup_{\mathbf{w}_N \in X_N(\Omega)} \frac{b(\mathbf{w}_N, q_N)}{\|\mathbf{w}_N\|_{H(\mathrm{div},\Omega)}} \geqslant \beta' \|q_N\|_{L^2(\Omega)}. \tag{3.28}$$

The constant $\beta' > 0$ does not depend on N.

The implementation of the discrete variational problem (3.26), (3.27) induces a symmetric algebraic system:

$$M_N \mathbf{U} + D_N \mathbf{\Psi} = \mathbf{f}, \tag{3.29}$$

$$D_N^T \mathbf{u} = 0. \tag{3.30}$$

The vector $\mathbf{u}$ contains the velocity degrees of freedom on the staggered grids, while $\mathbf{\Psi}$ represents the discrete values of ψ_N on GL grid. The diagonal mass matrix M_N is associated with the discrete inner product

$$(\mathbf{u}_N, \mathbf{w}_N)_N^{GL} = \sum_{i=0}^{N} \sum_{j=1}^{N} (u_x w_x)(\xi_i, \tau_j)\rho_i \omega_j + \sum_{i=1}^{N} \sum_{j=0}^{N} (u_y w_y)(\tau_i, \xi_j)\omega_i \rho_j. \tag{3.31}$$

The rectangular matrix D_N corresponds to the discretization of the variational form $(\nabla \cdot \mathbf{v}_N, q_N)_N^G$. In (3.29) $\mathbf{f}$ represents the quantity $(\mathbf{u}_N^*, \mathbf{w}_N)_N^{GL}$.

The algebraic system (3.29), (3.30) is solved using the Uzawa algorithm: One eliminates the velocity from (3.29) that is then inserted in (3.30)

$$D_N^T M_N^{-1} D_N \mathbf{\Psi} = D_N^T M_N^{-1} \mathbf{f}.$$

The existence of the solution of the algebraic square system (3.25) is ensured by the fact that the second member is in the range of the discrete operator. This property must be numerically checked before any resolution. In addition one can verify that the image is orthogonal to the kernel and thus their intersection is reduced to the null vector. The latter property makes it possible to ensure the uniqueness of the solution at least when the system is solved by an iterative method of the Krylov type (Conjugate Gradient in our case) while starting the iterative algorithm by an initial guess in the range (zero for example).

4. Numerical results

To illustrate the equivalence and compare the efficiency of the two approaches of the Helmholtz decomposition, we have carried out numerical experiments in the square $\Omega =]-1,+1[^2$ assessing the accuracy of the two methods. As example we studied the case $\mathbf{u}^* = \mathbf{v} - \nabla\psi$ with:

$$\mathbf{v}(x,y) = (-\sin(\pi x)\cos(\pi y), \cos(\pi x)\sin(\pi y)),$$

$$\psi = -\sin(\pi(x+y)).$$

The two components $\mathbf{v}$ and ψ are approximated respectively by $\mathbf{v}_N$ and ψ_N. As is well known, the spectral approximation error of analytical functions converges exponentially towards zero as ρ^N where N is the polynomial degree and $\rho \in]0,1[$ (see [4]). We expect the same error behavior in the present case.

Fig. 1 exhibits the computation for the primal formulation (3.21). On a semi-logarithmic scale for the L^2 error as a function of the polynomial degree N, one observes the typical spectral decay of the error $\|\mathbf{v}-\mathbf{v}_N\|_{(L^2(\Omega))^2}$ (see circles), and of $\|\mathrm{curl}(\mathbf{u}_\mathbf{N}^*-\mathbf{v}_\mathbf{N})\|_{L^2(\Omega)}$ (see squares), while $\|\mathrm{div}\,\mathbf{v}_N\|_{L^2(\Omega)}$ (see triangles) is nearly zero machine.

Fig. 2 gives the same information as Fig. 1 except for the fact that the various quantities have been computed with the dual formulation (3.26) and (3.27). The numerical results are quite close to those displayed on Fig. 1.

The method we used to solve the algebraic system (3.25) is the Conjugate Gradient (CG) algorithm. Table 1 gives the number of iterations needed to converge up to 10^{-14}. This number remains low which makes the primal approach attractive.

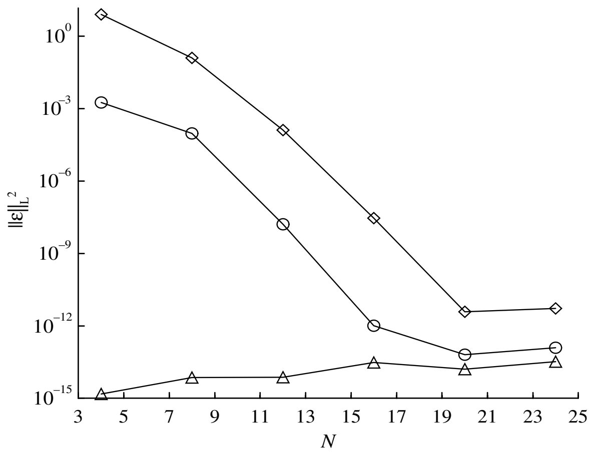

Fig. 1. Semi-logarithmic plot for the L^2 error as a function of N using the primal formulation. $\bigcirc := \|\mathbf{v}-\mathbf{v}_N\|_{(L^2(\Omega))^2}$, $\Diamond := \|\mathrm{curl}(\mathbf{u}_N^*-\mathbf{v}_N)\|_{L^2(\Omega)}$ and $\triangle := \|\mathrm{div}\,\mathbf{v}_N\|_{L^2(\Omega)}$.

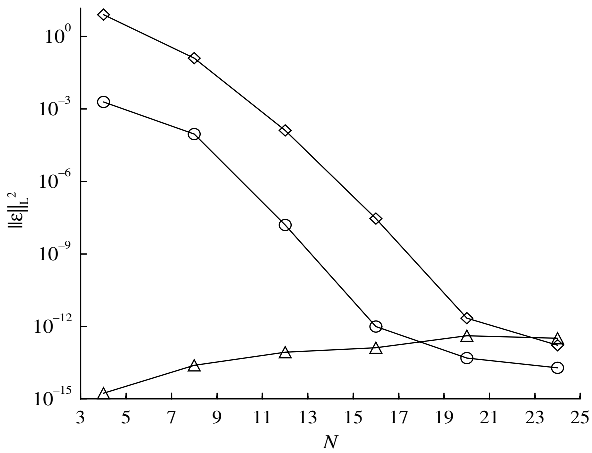

Fig. 2. Semi-logarithmic plot for the L^2 error as a function of N using the dual formulation. $\bigcirc := \|\mathbf{v}-\mathbf{v}_N\|_{(L^2(\Omega))^2}$, $\Diamond := \|\mathrm{curl}(\mathbf{u}_N^*-\mathbf{v}_N)\|_{L^2(\Omega)}$ and $\triangle := \|\mathrm{div}\,\mathbf{v}_N\|_{L^2(\Omega)}$.

Table 1
Number of iterations used by CG to solve (3.25)

N	4	8	12	16	20	24
CG	2	7	7	7	4	9

5. Conclusion

We have shown that the **grad**(div) operator offers an efficient tool for the Helmholtz decomposition of a vector field. We have proposed two ways to carry out this decomposition: We can use either the primal formulation or the dual one. The calculations performed on analytical functions for both formulations give similar results and clearly show the accuracy of the method.

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